Mathematics 1A, Fall 2010 — M. Christ — Practice Final Examination

Formula List

$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\cos(x) = -\sin(x)$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\frac{d}{dx}\sec(x) = \tan(x)\sec(x)$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$	$\frac{d}{dx}\csc(x) = -\cot(x)\csc(x)$
$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arctan(x) = (1+x^2)^{-1}$
$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$	Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
Volume of a sphere: $\frac{4}{3}\pi r^3$	Volume of a right circular cone: $\frac{1}{3}\pi r^2 h$

(1) *(3 points) for each part.* Calculate the following. Show your steps in an organized fashion, and place final answers in $\boxed{\text{boxes}}$.

(1a) The equation of the line tangent to $f(x) = x + e^x$ at $x = 2$.

(1b) $\frac{d}{dx}\sqrt{3 + \ln(\ln(x))}$.

(1c) $\lim_{x\to\pi/2} \frac{\cos(x)}{x-\pi/2}$

(1d) $\frac{d}{dx}x^{\cos(x)}$. (Here $x > 0$.)

(1e) $\int \frac{d}{dx}\sqrt{|\sin(x) + \cos(x)|}\, dx$.

(1f) $\frac{d}{dx}\int_0^{\sin(2x)} \arcsin(t)\, dt$.

(1g) $\int \sin(x)\cos(x)\, dx$

(2a) *(3 points)* $\int_0^1 (1-x^2)^{-1/2}\, dx$

(2b) *(3 points)* $\sum_{i=1}^4 i^2 \cos(\pi i)$

(2c) *(3 points)* $\int_{-2}^2 x\ln(1+x^4)\, dx$

(2d) *(4 points)* $\int (1-x^2)^{-3/2}\, dx$ (You need not simplify your answer.)

(2e) *(4 points)* $\lim_{x\to\infty}\left((x + x^{1/3})^{2/3} - x^{2/3}\right)$

(2f) *(4 points)* Express an approximation to $\int_1^3 e^{x^2}\, dx$ as a right endpoint Riemann sum with $n = 3$. Your answer need not be simplified; it could be expressed as a sum of several numbers.

(2g) *(4 points)* $\lim_{n\to\infty}\sum_{i=1}^n \frac{n}{n^2+i^2}$. (Either use a method taught in this course, or justify your steps in full detail.)

(2h) *(4 points)* Use Newton's method with initial approximation $x_1 = 10$ and one step to approximate the cube root of 996.

(3) *(6 points)* A right circular cone has height h and has a circular base of radius r. Its volume is $\frac{1}{3}\pi r^2 h$. Suppose that $r^2 + h^2 = 1$. For what value of h is the volume of the cone maximized? What is the maximum volume?

(4) *(9 points)* Sketch a graph of the function $f(x) = 4 + xe^{-1/2x}$. Indicate any horizontal or vertical asymptotes, but you need not indicate any slant asymptotes. Indicate intervals on which f is increasing and decreasing, local maxima and minima, inflection points, and intervals on which the graph is concave up or down. It is not possible to calculate intercepts exactly. Instead, determine exactly how many intercepts there are, and indicate roughly where they are located. You may use the formulas $f'(x) = (1 + \frac{1}{2}x^{-1})e^{-1/2x}$ and $f''(x) = \frac{1}{4}x^{-3}e^{-1/2x}$.

(5) Newton's law of cooling says: The rate of cooling of a body is proportional to the difference between that body's temperature, and the temperature of its environment. In a cafe where the ambient room temperature is a steady 70 degrees, a cup of coffee is served at 190 degrees. (All temperatures are measured in degrees Fahrenheit.) Assume that Newton's law of cooling applies.

(5a) *(3 points)* Let $f(t)$ be the temperature of the coffee at time t. Write a differential equation satisfied by $f(t)$. Your equation may include one or more unknown constants.

(5b) *(2 points)* Write the general solution of your differential equation.

(5c) *(3 points)* 3 minutes after the coffee is served, its temperature is 180 degrees. At what time will the coffee cool to 160 degrees? Express your answer in minutes after the coffee is served.

(6) Show your steps in an organized, legible manner to receive credit. Let $\mathcal{C}$ be the circle with radius 1 centered at $(2,0)$ in the xy plane. The region enclosed by $\mathcal{C}$ is revolved around the y axis to generate a three dimensional solid known to mathematicians as a torus, and to law enforcement officers as a staple food.

(6a) *(3 points)* Using the method of cylindrical shells, express the volume of this solid as an integral.

(6b) *(6 points)* Evaluate the integral in **(6a)** *using methods and results taught in this course.* (You might be unable to find an antiderivative, but it is possible to evaluate the *definite* integral using material taught in the course.)

(7) Short answer questions. *2 points for each part. These questions require only brief answers, and require little or no calculation.*

(7a) An emu and a wombat race along a straight line, beginning at time $t = 0$. The wombat is given a head start. At time t, their positions are $E(t)$ and $W(t)$, respectively. Suppose that $W(t) = 1 + t$ for all $t \geq 0$, that $E(0) = 0$, and that $E''(t) < 0$ for all $t \geq 0$. What is the maximum possible number of times $t > 0$ at which $E(t)$ can be equal to $W(t)$? Explain your answer briefly.

(7b) Define: f has an inflection point at x.

(7c) Let $t =$ time, $s(t) =$ the position of a projectile at time t, and $v(t) =$ its velocity. Let $a < b$ be two times. We have defined two kinds of averages of v over the interval $[a, b]$: (i) The net change in position divided by the elapsed time, and (ii) $(b - a)^{-1} \int_a^b v(t)\, dt$. How are these two averages related to one another? Explain very briefly.

(7d) If $f(0) = 0$, $f'(0) = -1$, and $f''(x) \leq 2$ for all x, what is the largest possible value of $f(3)$?

(7e) If $f'(-1) = f'(1)$, and if $f''(x)$ exists and is a continuous function on $[-1, 1]$, then two conclusions can be drawn about f''. What are they?

(7f) If f and its derivative f' are continuous functions defined for all real numbers, and if $f'(x + 1) = f'(x)$ for all x, what conclusion can be drawn about f?

(7g) Let f be a continuous function defined for all $x \geq 0$. For $s > 0$ let $g(s) = s^{-1} \int_0^s f(x)\, dx$. Suppose that t is a positive number such that $g(t) \geq g(s)$ for every $s > 0$. Find an equation relating $f(t)$ to $g(t)$.

(7h) Explain briefly how the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ in Newton's method is derived.

(7i) If you were asked to derive the formula $\operatorname{arcsec}'(x) = \frac{1}{x\sqrt{x^2-1}}$, assuming that arcsec is differentiable, how would you begin? You need not give a proof or do any calculations, but show that you know what to calculate.